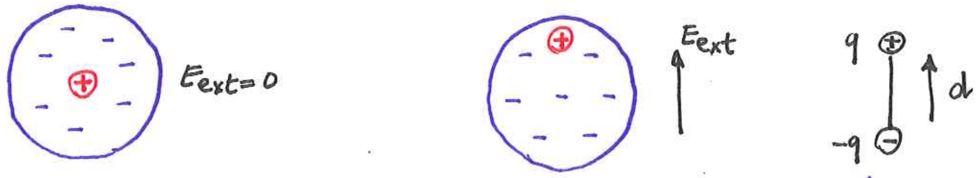
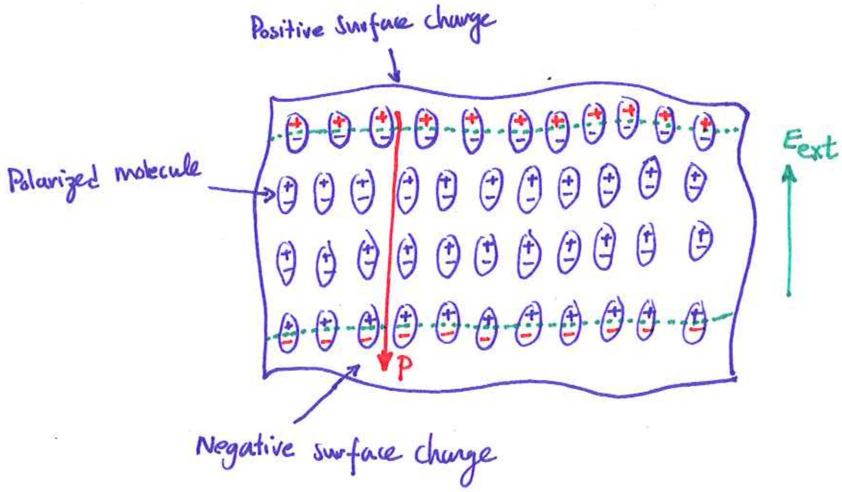
Dielectrics

When a dielectric is under applied electric field E_{ext}, the electrons cannot move inside the material, but they can be shifted around the nucleus. This polarizes the atoms or molecules in the material.



The charge separation induces an internal electric field that is smaller than E_{ext}.

The induced internal electric field is called polarization field and opposes E_{ext}. Consequently, the net electric field in the dielectric material is smaller than E_{ext}.



Whereas $\vec{D}$ and $\vec{E}$ are related by ε_0 in free space, at presence of microscopic dipoles in the dielectric material alters that relationship in material to:

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

Electric Polarization field

Linear dielectric: $\vec{P} \propto \vec{E}$ linearly

Isotropic dielectric: $\vec{P}$ doesn't depend on direction

Anisotropic dielectric: $\vec{E}$ and $\vec{D}$ are in different direction due to $\vec{P}$

In linear, isotropic, and homogeneous material we can write:

$$\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}}$$

χ_e → electric susceptibility

$$\rightarrow \vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1+\chi_e) \vec{E} = \varepsilon \vec{E}$$

$$\rightarrow \varepsilon = \varepsilon_0 (1+\chi_e) \rightarrow \boxed{\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1+\chi_e}$$

For most conductors $\chi_e \cong 0 \rightarrow \varepsilon_r \cong 1$

For air $\varepsilon_r \simeq 1.0006$ at sea level.

In $\vec{E}$ is very strong, it may free the electrons completely from the molecules and create current. This happens with sparking and permanent damage. This field is called dielectric strength and the abrupt chang in behavior is called a dielectric breakdown.

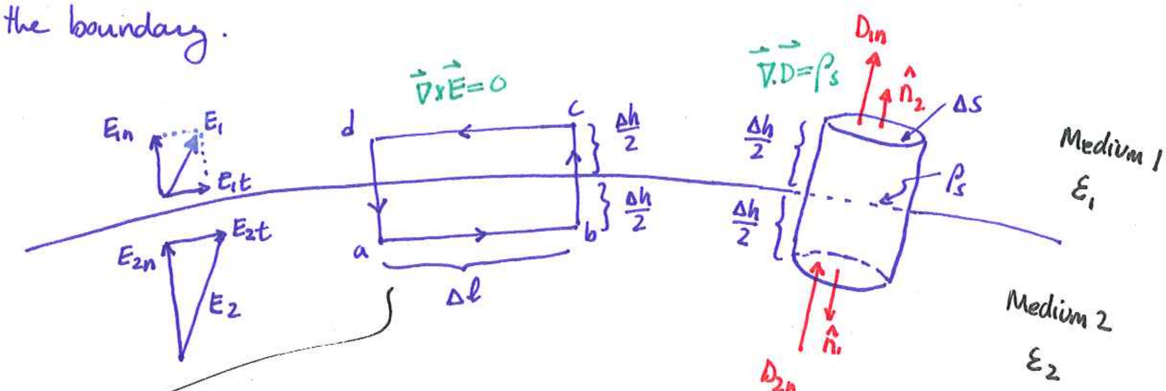
The dielectric strength E_{ds} for air is $= 3 \frac{MV}{m}$ → obseved is thundercloud.

E_{ds} for glass $= 25-40 \frac{MV}{m}$

for Mica $= 200 \frac{MV}{m}$

Electric Boundary Conditions

The electric field at interface of two dissimilar media may be discontinous if surface charge exists at the boundary.



$$\oint_C \vec{E} \cdot \vec{dl} = \int_a^b \vec{E}_2 \cdot \vec{dl} + \underbrace{\int_b^c \vec{E} \cdot \vec{dl}}_{\substack{=0 \\ \Delta h \to 0}} + \int_c^d \vec{E}_1 \cdot \vec{dl} + \underbrace{\int_d^a \vec{E} \cdot \vec{dl}}_{\substack{=0 \\ \Delta h \to 0}} = 0$$

$$\left\{\begin{array}{l}\vec{E}_1=\vec{E}_{1t}+\vec{E}_{1n}\\ \vec{E}_2=\vec{E}_{2t}+\vec{E}_{2n}\end{array}\right. \rightarrow \int_a^b \vec{E}_2\cdot d\vec{l} = E_{2t}\,\Delta l \quad \text{and} \quad \int_c^d \vec{E}_1\cdot d\vec{l} = -E_{1t}\,\Delta l$$

$$\rightarrow \quad E_{2t}\Delta l - E_{1t}\Delta l = 0 \implies \boxed{E_{1t}=E_{2t}}$$

The tangential component of the electric field is continous across the boundary between any two media.

The boundary condition for tangential component of the electric flux density is:

$$\vec{D}=\varepsilon\vec{E} \rightarrow \boxed{\frac{D_{1t}}{\varepsilon_1}=\frac{D_{2t}}{\varepsilon_2}}$$

Next, we apply Gauss's law to find the relation between the normal components:

$$\oint_s \vec{D}\cdot d\vec{s} = Q \rightarrow \oint_s \vec{D}\cdot d\vec{s} = \int_{top} \vec{D}_1\cdot\hat{n}_2\,ds + \underbrace{\int_{side} \vec{D}\cdot d\vec{s}}_{\substack{=0\\ \Delta h\to 0}} + \int_{bottom} \vec{D}_2\cdot\hat{n}_1\,ds = Q$$

$$\left.\begin{array}{l}\Delta Q=\rho_s\,\Delta S\\ \Delta h\to 0\\ \hat{n}_1=-\hat{n}_2\end{array}\right\} \int \vec{D}_1\cdot\hat{n}_2\,ds - \int \vec{D}_2\cdot\hat{n}_2\,ds = \int(\vec{D}_1-\vec{D}_2)\cdot\hat{n}_2\,ds = \int\rho_s\,ds$$

$$\rightarrow \boxed{\hat{n}_2\cdot(\vec{D}_1-\vec{D}_2)=\rho_s \quad (C/m^2)}$$

$$\left.\begin{array}{l}\hat{n}_2\cdot\vec{D}_1=D_{1n}\\ \hat{n}_2\cdot\vec{D}_2=D_{2n}\end{array}\right\} \boxed{D_{1n}-D_{2n}=\rho_s}$$

The normal component of $\vec{D}$ changes abruptly at the boundary when there exists surface charge at the boundary.

For $\vec{E}$ at boundary we have: $$\boxed{\varepsilon_1 E_{1n}-\varepsilon_2 E_{2n}=\rho_s}$$

In summary the conservative property of $\vec{E}$:

$$\nabla \times E = 0 \iff \oint_C \vec{E} \cdot \vec{dl} = 0$$

results that $\vec{E}$ has a continous tangential component across a boundary.

And the divergence property of $\vec{D}$:

$$\nabla \cdot D = \rho_v \iff \oint_S \vec{D} \cdot \vec{ds} = Q$$

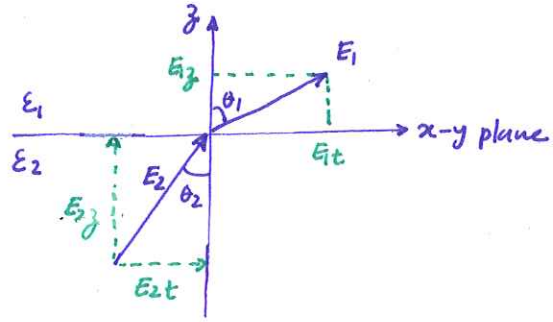
led to the result that normal component of $\vec{D}$ changes by ρ_s across the boundary.

Example

The x-y plane is a charge free boundary as shown in the picture.

If $\vec{E}_1 = \hat{x} E_{1x} + \hat{y} E_{1y} + \hat{z} E_{1z}$, find $\vec{E}_2$ and the angles θ_1 and θ_2.

$$E_{1t} = E_{2t} \rightarrow \begin{cases} E_{1x} = E_{2x} \\ E_{1y} = E_{2y} \end{cases}$$



Since $\rho_s = 0 \rightarrow D_{2z} = D_{1z} \rightarrow \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}$

$$\rightarrow \vec{E}_2 = \hat{x} E_{1x} + \hat{y} E_{1y} + \hat{z} \frac{\varepsilon_1}{\varepsilon_2} E_{1z}$$

For the angles we can write:

$$\tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}}$$

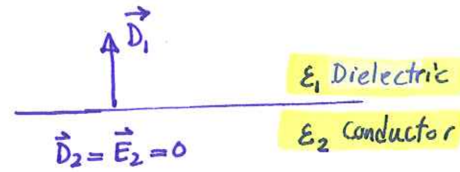
$$\tan\theta_2 = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{\left(\frac{\varepsilon_1}{\varepsilon_2}\right) E_{1z}}$$

$$\rightarrow \frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

Dielectric - Conductor Boundary

If medium 1 is a dielectric and medium 2 a perfect conductor, in the perfect conductor:

$E_2 = D_2 = 0$ everywhere in the conductor.



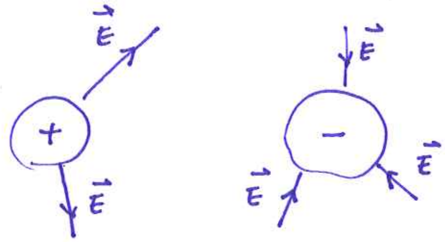
$\Rightarrow \quad E_{1t} = E_{2t} = 0$

$D_{1n} - D_{2n} = \rho_s \rightarrow D_{1n} = \varepsilon_1 E_{1n} = \rho_s$

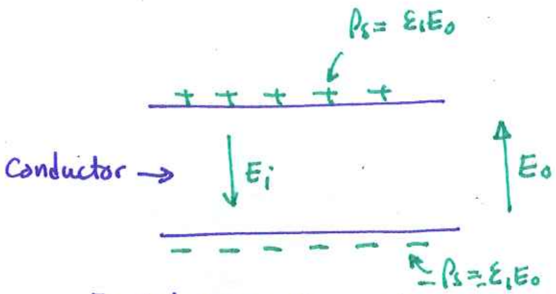
$\rightarrow \quad \vec{D}_1 = \varepsilon \vec{E}_1 = \hat{n} \rho_s$ (at conductor surface).

$\hat{n}$ is a unit vector normal & outward from the conductor surface.

So if $\rho_s > 0$, the electric field is outward and if $\rho_s < 0$, it is inward.



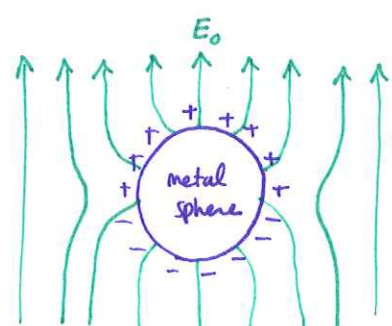
Now consider two situation: a conducting slab in external field E_o and a metal sphere in an external electric field:



E_o induces surface charge.

surface charges induce an internal field E_i. Since E inside the conductor is zero $\Rightarrow$

$\vec{E}_o + \vec{E}_i = 0 \Rightarrow \vec{E}_i = -\vec{E}_o$



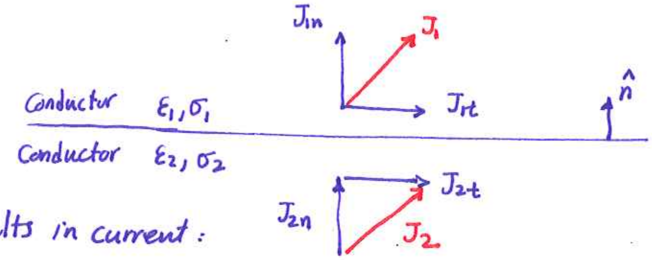
E_o induces charges on the sphere so that positive charges are accumulated on one side and negative charges on the other side. The field resulted from the surface charges bends the field so that $\vec{E}$ is normal to the surface.

Conductor- Conductor Boundary

We now look at the boundary of two perfect conductors:

$E_{1t} = E_{2t}$

$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$



Since the media is conductive, electric field results in current:

$$\vec{J} = \sigma \vec{E} \quad \Rightarrow \quad \vec{E} = \frac{\vec{J}}{\sigma} \quad \Rightarrow \quad \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \quad \text{and} \quad \varepsilon_1 \frac{J_{1n}}{\sigma_1} - \varepsilon_2 \frac{J_{2n}}{\sigma_2} = \rho_s$$

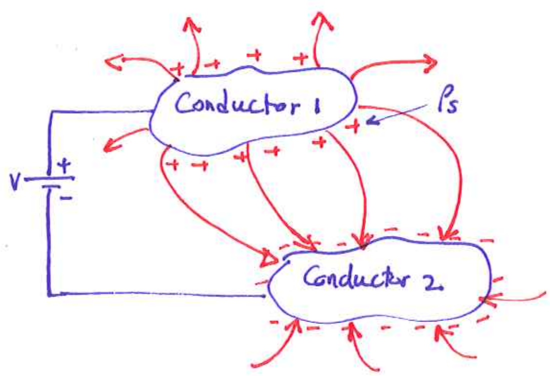
However J_{1n} cannot be different from J_{2n} otherwise the number of charges arriving at interface will be different from the number of charges leaving the surface and violates the conservation of charge. Hence:

$$J_{1n} = J_{2n} \rightarrow \boxed{J_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s \quad \text{(electrostatics)}}$$

Capacitance

A capacitor is formed when two conductors are separated by an insulator (dielectric).

If a d-c voltage is applied to the conductors, charge of equal and opposite polarity is transfered to the conductors' surface, $+Q$ and $-Q$ in the figure. The charge is distributed on the surface of the conductors so that the electric field inside the conductor remains zero (i.e. equipotential surface).



The capacitance is defined as:

$$\boxed{C = \frac{Q}{V}} \ (F)$$

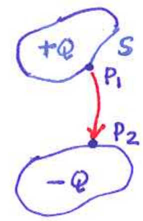
The tangential component of $\vec{E}$ on the surface is zero and $\vec{E}$ is normal to the surface everywhere.

Since $\vec{E}$ inside the metal is zero, we can write:

$$D_n^{\text{outside}} - \underbrace{D_n^{\text{inside}}}_{=0} = \rho_s \implies E_n = \frac{\rho_s}{\varepsilon} \quad \text{(at conductor surface)}$$

ρ_s is the surface charge density. So:

$$\left.\begin{aligned} Q &= \int_S \rho_s\, ds = \int_S \varepsilon \overbrace{\hat{n}\cdot\vec{E}}^{E_n}\, ds \\ V &= V_{12} = -\int_{P_2}^{P_1} \vec{E}\cdot\vec{dl} \end{aligned}\right\} \quad \boxed{C = \frac{\int_S \varepsilon \vec{E}\cdot\vec{ds}}{-\int_l E\cdot dl}}$$



Always take surface S on +Q and point P_1 on surface S. Also C is always independent from the strength of $\vec{E}$ as it appears both on numerator and denumerator. If the dielectric material in between is not a perfect insulator, then the current can flow between the conductors and the material has a finite resistance R:

$$R = \frac{-\int_l \vec{E}\cdot\vec{dl}}{\int_S \sigma \vec{E}\cdot\vec{ds}}$$

For a medium with uniform σ and ε, RC becomes:

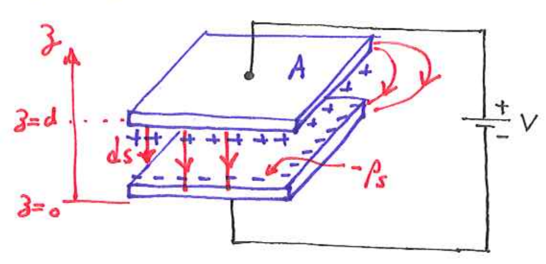
$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$ So we can find R, if we know C, or vice versa!

<u>Example</u> Capacitance and Breakdown Voltage of Parallel_Plate Capacitor:

Find the capacitance and beakdown voltage of a parallel-plate capacitor with plate area of A separated by d. The dielectric material is quartz and d=1 cm.

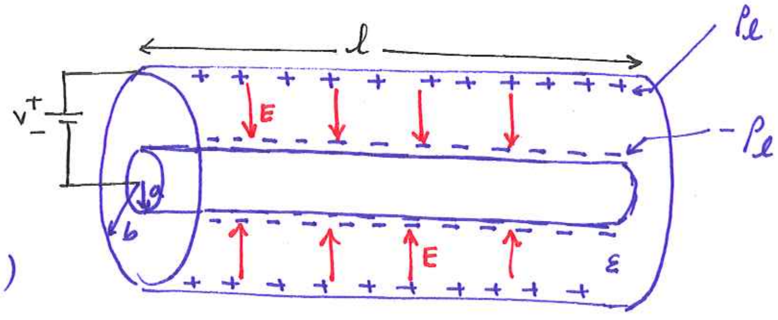
$$\left.\begin{aligned} &\vec{E} = -\hat{z}E \rightarrow V = -\int_0^d -E\hat{z}\cdot\hat{z}\,dz = Ed \\ &E = \frac{\rho_s}{\varepsilon} = \frac{Q/A}{\varepsilon} \rightarrow Q = \varepsilon A E \end{aligned}\right\} C = \frac{Q}{V} = \frac{\varepsilon A E}{d E}$$

$$\rightarrow \boxed{C = \varepsilon \frac{A}{d}}$$



Using Table 4-2, $E_{ds} = 30$ MV/m for quartz $\Rightarrow$ $V_{br} = E_{ds} d = 30 \times 10^{6} \times 10^{-2} = 3 \times 10^{5}$ V

Example Find the capacitance of a Coaxial line.



$$\rho_l = \frac{Q}{l}$$

$$E = -\hat{r}\frac{\rho_l}{2\pi\varepsilon r} = -\hat{r}\frac{Q}{2\pi\varepsilon r l}$$

$$V = -\int_a^b E \cdot dl = -\int_a^b \left(-\hat{r}\frac{Q}{2\pi\varepsilon r l}\right)\cdot(\hat{r}\,dr)$$

$$V = \frac{Q}{2\pi\varepsilon l}\ln\left(\frac{b}{a}\right) \rightarrow \boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln\frac{b}{a}}} \rightarrow C' = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \quad (\text{F/m})$$

Electrostatic Potential Energy

When a capacitor is charged the energy is stored in the form of electrostatic potential energy in the dielectric medium. The amount of energy W_e is related to Q, C, V:

$$dW_e = V\,dq \quad \text{from the basic definition of electric potential } V.$$

$$= \frac{q}{C}\,dq$$

$$\rightarrow W_e = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}$$

Since $C = \frac{Q}{V} \rightarrow \boxed{W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}}$ (J)

For the parallel-plate capacitor this is:

$$\left.\begin{array}{l} C = \varepsilon\frac{A}{d} \\ V = Ed \end{array}\right\} W_e = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon\frac{A}{d}(Ed)^2 = \frac{1}{2}\varepsilon E^2 \underbrace{Ad}_{\mathcal{V}}$$

$\mathcal{V}$ (volume of capacitor)

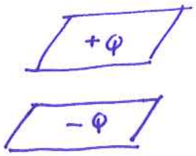
$\rightarrow$ Electrostatic energy density w_e is: $\boxed{w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\varepsilon E^2}$ (J/m^3)

stored energy per unit volume of the capacitor

We can generalize this to any volume v containing a dielectric ε:

$$\boxed{W_e = \frac{1}{2}\int_v \varepsilon E^2 \, dv} \quad (\mathrm{J})$$

If we consider the force F acting on the parallel plates due to the opposite charge attraction from the plates:



$$dW = \vec{F}\cdot\vec{dl}$$

This work reduces the energy stored in the capacitor. So: $dW = -dW_e$

We may write dW_e in gradient form:

$$dW_e = \vec{\nabla}W_e \cdot \vec{dl} \Rightarrow \qquad dW = \vec{F}\cdot\vec{dl} = -dW_e = \vec{\nabla}W_e\cdot\vec{dl} \Rightarrow$$

$$\boxed{\vec{F} = -\vec{\nabla}W_e} \quad (\mathrm{N})$$ this correct only if the charges in the system are constant.

__Example__ Calculate the force between the plates of a two parallel plate capacitor.

To apply this equation to a parallel-plate capacitor, we have:

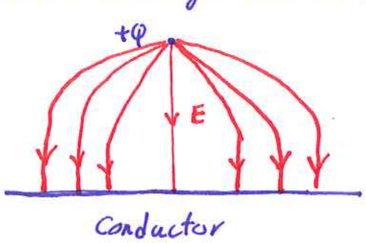
$$W_e = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2 z}{2\varepsilon A} \qquad \left(C = \varepsilon\frac{A}{z}\right)$$

$$\vec{F} = -\vec{\nabla}W_e = -\hat{z}\frac{\partial}{\partial z}\left(\frac{Q^2 z}{2\varepsilon A}\right) = -\hat{z}\left(\frac{Q^2}{2\varepsilon A}\right)$$

Since $Q = \varepsilon A E \rightarrow \quad \vec{F} = -\hat{z}\,\dfrac{\varepsilon A E^2}{2}$ (parallel-plate capacitor)

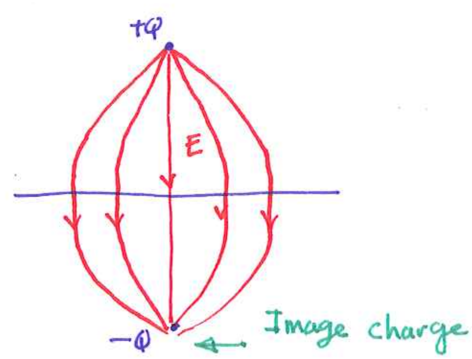
Image Method

Consider a charge next to a conductor:



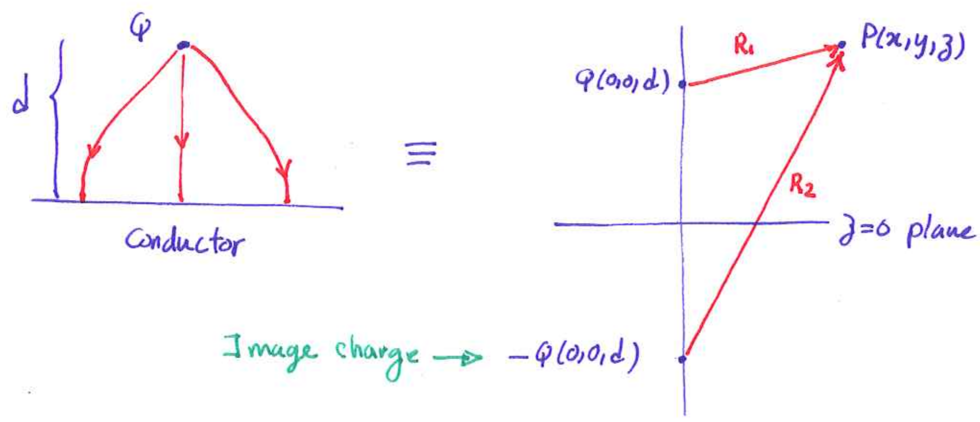
$$\equiv$$

Since the electric field is normal to the surface of the conductor, we may model this problem with two opposite charges as:



Example

Use image method to determine V and $\vec{E}$ at an arbitrary point $P(x,y,z)$ in the $z>0$ due to a charge Q in free space at a distance d above a ground conducting plane:



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R_1^3}\vec{R_1} + \frac{1}{4\pi\varepsilon_0}\frac{-Q}{R_2^3}\vec{R_2}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{x\hat{x}+y\hat{y}+(z-d)\hat{z}}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{x\hat{x}+y\hat{y}+(z+d)\hat{z}}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \quad \text{for } z \geq 0$$

$\vec{E}=0$ for $z<0$ since E is zero in side the conductor.